Introduction to Lagrangian Mechanics

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1 Basics

Most of us learn physics in much the same way Newton described it. In particular, Newtonian mechanics has a specific process to solve problems: find the forces, get the accelerations, integrate twice. And now you have the laws of motion. This isn't, however, the only was to think of physics. Lagrangian mechanics focuses instead on energy, and we'll be describing this perspective on mechanics here.

A warning: Lagrangian mechanics, much like any Newtonian mechanics worth doing, requires a deep, loving friendship with calculus, differential equations and all. This is assumed of the reader.

So, let's begin. We'll be talking a lot here about kinetic and potential energy. Now, it would be nice if we'd call them K and P, but physicists are weird, so we'll refer to them as T and U. These are really functions of three variables: time (t), position (q), and velocity ($\dot{q}$). Yes, we'll be using physics notation, where a dot over a variable represents the time derivative of that variable.

Now, we define a quantity $L = T - U$, or $L(t, q, \dot{q}) = T(t, q, \dot{q}) - U(t, q, \dot{q})$. The fundamental law we'll be using is Lagrange's Law, which states that

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}}L = \frac{\partial}{\partial q}L$$

We won't be proving this result, but if you're interested, you can find the standard derivation on Wikipedia.

So now, we have a new procedure for solving physics problems. We first find the kinetic and potential energies of all of the particles. Then, we can use Lagrange's Law, along any of the variables, to find a differential equation. Then we solve that equation with a mix of voodoo and advanced math. And then you have the equations of motion.

2 A Simple Example

Let's consider an object falling in a uniform gravitational field. We all know what the answer should be: $q = \frac{g}{2}t^2$. So let's see how we get that with Lagrangian mechanics.

Now, the kinetic energy is obvious. We all know that $KE = \frac{1}{2}mv^2$, which we're going to write as $T = \frac{1}{2}m\dot{q}^2$. The potential energy, we know, is equal to weight times height, or $U = mgq$. So:

$$\begin{aligned} L &= \frac{1}{2}m\dot{q}^2 - mgq \\ \frac{\partial L}{\partial q} &= -mg \\ \frac{\partial L}{\partial \dot{q}} &= m\dot{q} \\ \frac{d}{dt}m\dot{q} &= m\ddot{q} \\ m\ddot{q} &= -mg \\ \ddot{q} &= -g \end{aligned}$$

So this shows that the acceleration of an object in a uniform gravitational field is g, downwards. We can, of course, integrate twice to obtain the law we all know and love.

By the way, if we know that $T = \frac{1}{2}m\dot{q}^2$ and if U is independent of q, we can generalize this to:

$$m\ddot{q} = F = -\frac{\partial U}{\partial q}$$

This is, of course, just a restatement of Newton's second law.

3 An Intuitive Sense of L

We, being physicists, don't like the formality of the above description. Sure, the math seems to work, but what about an intuitive sense of the Lagrangian?

Well, we know that the total energy of a system is constant. And we know that the kinetic energy is the total amount of stuff happening, whereas potential energy is what *could* happen. So since $L = T - U$, the Lagrangian measures, in a sense, the total *liveliness* of a system.

In fact, a proper derivation of Lagrangian mechanics would focus on the action, which is defined as the integral of L over the span of time you care about. A fundamental principle says that the path taken by a particle is the one with the least action (this is the Principle of Least Action, which in special cases is the Principle of Least Time in optics). What does this mean? Well, if the Lagrangian is the liveliness of a system, the action is the total amount happening. So if nature seeks to minimize the total amount happening, what does this mean? It means that *nature is lazy.*

Think, for example, of a ball that is moving under gravity. It is the least lively at the top, so it spends a lot of time there. But if it spends too much time up there, it needs to be more lively when going up. So the parabola is a compromise between the action in the beginning and the action at the end.

4 Why use Lagrangian Mechanics?

The basic difference between Newtonian mechanics and Lagrangian mechanics is the focus on forces as opposed to energy. At first, this might strike one as a more or less meaningless distinction. But there is an important distinction: force is a vector, energy a scalar. The benefit here is that force will change in different coordinate systems, but the Lagrangian stays the same. Thus, while Newtonian mechanics works well in rectangular coordinates, it can't be used in any other coordinate system. This is a pity, because many things, such as rotation, live in other coordinate systems. If you're considering planar rotation, you almost certainly want to work in polar coordinates. If 3D rotation is involved, you want to work in spherical coordinates. And for that you want Lagrangian mechanics. In this vein, the next section employs Lagrangian mechanics in the context of a rotating coordinate system.

5 Rotational Motion

Imagine that you are in a giant rotating drum. You can move about in three dimensions at will, but, unbeknownst to you, the drum is constantly spinning with an angular velocity ω. What we want to do is to see how physics would look from this rotating standpoint. We'll denote the position in the drum as $\overline{q}$ and the real-world position (relative to some fixed coordinate system) as q.

Using some trigonometry, we know that

$$
\begin{aligned}
q_x &= \overline{q}_x \cos \omega t - \overline{q}_y \sin \omega t \\
q_y &= \overline{q}_y \cos \omega t + \overline{q}_x \sin \omega t \\
q_z &= \overline{q}_z
\end{aligned}
$$

We can differentiate this term-by-term to find $\dot{q}$.

$$
\begin{aligned}
\dot{q}_y &= (\dot{\overline{q}}_x - \omega \overline{q}_y) \cos \omega t - (\dot{\overline{q}}_y + \omega \overline{q}_x) \sin \omega t \\
\dot{q}_y &= (\dot{\overline{q}}_y + \omega \overline{q}_x) \cos \omega t + (\dot{\overline{q}}_x - \omega \overline{q}_y) \sin \omega t \\
\dot{q}_z &= \dot{\overline{q}}_z
\end{aligned}
$$

Now, we'll make things easy and assume that there are no masses or magnets involved, so we'll set $U = 0$. Then:

$$
\begin{aligned}
L &= \frac{1}{2} m \dot{q}^2 \\
&= \frac{1}{2} m (\dot{q}_x^2 + \dot{q}_y^2 + \dot{q}_z^2) \\
&= \frac{1}{2} m \left((\dot{\bar{q}}_x - \omega \bar{q}_y)^2 + (\dot{\bar{q}}_y + \omega \bar{q}_x)^2 + \dot{\bar{q}}_z^2 \right)
\end{aligned}
$$

Note that L does not depend on t. This is good, because we expect that everything should be the same no matter when we start the clock. Now, this equation looks like a rather nice formula, and it may even seem that this is as good as it gets, but there's one further simplification. Letting $\overline{\omega} = \langle 0, 0, \omega \rangle$:

$$
L = \frac{1}{2} m (\dot{\overline{q}} + \overline{\omega} \times \overline{q})^2
$$

If you've worked with rotation before, you know that $\overline{\omega}$ is the angular velocity *vector*, and all of this is not so surprising. But for the others, you can see that:

$$\bar{\omega} \times \bar{q} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & \omega \\ \bar{q}_x & \bar{q}_y & \bar{q}_z \end{vmatrix} = \omega \bar{q}_x \hat{\jmath} + (-\omega \bar{q}_y)\hat{\imath}$$

This is just what we needed above. From here, things are easy, using Lagrange's Law.

$$\begin{aligned}
\frac{1}{m}\frac{\partial L}{\partial \bar{q}} &= -\bar{\omega} \times (\dot{\bar{q}} + \bar{\omega} \times \bar{q}) \\
&= -\bar{\omega} \times \dot{\bar{q}} + (-\bar{\omega} \times (\bar{\omega} \times \bar{q})) \\
\frac{1}{m}\frac{\partial L}{\partial q} &= \dot{\bar{q}} + \bar{\omega} \times \bar{q} \\
\frac{d}{dt}\frac{1}{m}\frac{\partial L}{\partial q} &= \ddot{\bar{q}} + \bar{\omega} \times \dot{\bar{q}} \\
\ddot{\bar{q}} + \bar{\omega} \times \dot{\bar{q}} &= -\bar{\omega} \times \dot{\bar{q}} + (-\bar{\omega} \times (\bar{\omega} \times \bar{q})) \\
\ddot{\bar{q}} &= -2\bar{\omega} \times \dot{\bar{q}} - \bar{\omega} \times (\bar{\omega} \times \bar{q})
\end{aligned}$$

Lets try to figure this out term by term. First, let's take $-\bar{\omega} \times (\bar{\omega} \times \bar{q})$. If we split $\bar{q}$ into $\bar{q}_{\|}$ parallel to $\bar{\omega}$ and $\bar{q}_{\perp}$ perpendicular to it, we see that

$$-\bar{\omega} \times (\bar{\omega} \times (\bar{q}_{\|} + \bar{q}_{\perp})) = -\bar{\omega} \times (\bar{\omega} \times \bar{q}_{\|} + \bar{\omega} \times \bar{q}_{\perp})$$

Since the cross product of two parallel vectors is 0, we see that this equals

$$\bar{\omega}^2 \bar{q}_{\perp} = \omega^2 \bar{q}_{\perp}$$

This term, we see, is really just our centrifugal "force", really a psuedo-force, which always pushes us out from the center. We also see that it increases linearly with distance from the origin, and is proportional to the square of the angular velocity.

Now, what about that second term, $-2\bar{\omega} \times \dot{\bar{q}}$? Since it is always perpendicular to the velocity, we can easily recognize this as the Coriolis "force", which is what causes toilets to flush counterclockwise and all.

See, Lagrangian mechanics isn't all that bad!

6 Two-body Gravitational Interaction

Now, let's apply our new toy to something that is very annoying to do in Newtonian mechanics—gravitation. We'll have two bodies, located at r_1 and r_2, with masses m_1 and m_2.

$$
\begin{aligned}
T &= \frac{1}{2} m_1 \dot{r_1}^2 + \frac{1}{2} m_2 \dot{r_2}^2 \\
U &= k / \|r_1 - r_2\|
\end{aligned}
$$

Let's start to simplify this. First, we define the center of mass of the system, $R = \dfrac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$, and also r, the relative position, as $r_1 - r_2$. Then:

$$
\begin{aligned}
r_1 &= \frac{1}{2} \left(R + \frac{m_2}{m_1 + m_2} r \right) \\
r_2 &= \frac{1}{2} \left(R - \frac{m_1}{m_1 + m_2} r \right)
\end{aligned}
$$

We then get:

$$
L = \frac{1}{8}(m_1 + m_2)\dot{R}^2 + \frac{1}{8}\frac{m_1 m_2}{m_1 + m_2}\dot{r}^2 - \frac{k}{\|r\|}
$$

But this is really the L for the sum of two systems: one with T being the first term and with no U, and another for r, which contains the other two terms. Since there is no potential in the L for R, it is moving inertially, which means that the total momentum of the system (that is, the momentum of their center of mass) is conserved.

But what about the L for the system for r? Well, let's begin by factoring out (ignoring) the part about R. We'll also rename $\frac{1}{4}\frac{m_1 m_2}{m_1 + m_2}$ to μ. We then have that:

$$
L = \frac{1}{2}\mu \dot{r}^2 - \frac{k}{\|r\|}
$$

Using Lagrange's law, we now get that:

$$\frac{\partial L}{\partial r} = -\hat{r}\frac{\partial U}{\partial r}$$
$$\frac{\partial L}{\partial \dot{r}} = \mu\dot{r}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \mu\ddot{r}$$

Thus,

$$\mu\ddot{r} = -\hat{r}\frac{\partial U}{\partial r}$$

So since force is always along $\hat{r}$, r must stay within a plane (think about it—nothing ever pushes it out of that plane).

Now, let's make use of the fact that r lies within the plane and rewrite the above equation into polar form. Here, r really represents $\|r\|$:

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}) - \frac{k}{r}$$

Note that θ doesn't matter, only $\dot{\theta}$. This means that the system doesn't care if you rotate it (as it should be).

Now, we can use Lagrange's Law with respect to r or with respect to θ. Let's start with θ, as it is much simpler.

$$\frac{\partial L}{\partial \theta} = 0$$
$$\frac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta}$$

This means that $\mu r^2\dot{\theta}$ is constant. In fact, if we think about it, μr^2 is the moment of inertia, and $\dot{\theta}$ is the angular velocity. Thus, this mysterious quantity is really just the angular momentum, and so we've just shown that angular momentum is conserved (in this system). For the future, we'll call angular momentum (which is a constant) l.

Now, let's use Lagrange's Law with respect to r. First, we note that we can define $E = T + U = L + 2U$, the total energy of the system. Next, we can use the law of conservation of energy to see that E is constant. Further, we can use our discovery of the conservation of linear momentum to determine that $\dot{\theta} = \frac{l}{\mu r^2}$ So,

$$E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + U$$
$$\dot{r}^2 = 2\frac{E-U}{\mu} - \frac{l^2}{\mu^2 r^2}$$

Now, what is $\frac{dr}{d\theta}$? Well, remember that a dot over a variable means time derivative. So, we can use the chain rule, and we see that it is equal to $\frac{\dot{r}}{\dot{\theta}}$. So, we can get a differential equation of r in terms of θ:

$$\frac{\dot{r}}{\dot{\theta}} = \frac{dr}{d\theta} = r\sqrt{2(E-U)\frac{\mu r^2}{l^2} - 1} = \frac{r^2}{l}\sqrt{2E\mu + 2\frac{k\mu}{r} - \frac{l^2}{r^2}}$$

Don't worry about the algebra too much, by the way. Its most important that you know the general concepts. If there are any important shortcuts or tricks in the calculation, we'd point them out. Anyway, what do we do with this differential equation? Well, the solution is hairy, so we'll just give you the answer. How to get it... well, it's not super important. The basic solution we get is:

$$\frac{l^2}{\mu k}\frac{1}{r} = 1 + \cos\theta\sqrt{1 + \frac{2El^2}{\mu k^2}}$$

Letting $\alpha = \frac{l^2}{\mu k}$ and $\epsilon = \sqrt{1 + \frac{2El^2}{\mu k^2}}$, and rearranging a bit, we get:

$$r = \frac{\alpha}{1 + \epsilon\cos\theta}$$

We claim that this is the polar form of a conic section. Let's see why.

7 Polar Forms of Conic Sections

A conic section C can be defined as the set of all points P such the measure of PF is proportional to Pd, by a set constant ϵ, where F is called the focus and where the line d is called the directrix. Now, define a point B at the projection of P onto the line that passes through F and is perpendicular to

d (it may help you to draw a diagram). Now, if θ is the angle PFB, r is PF, and the distance Fd is k,

$$
\begin{aligned}
FB &= r\cos\theta \\
Pd &= FC - FB = k - r\cos\theta \\
FP &= r = \epsilon Pd \\
r &= \epsilon k - \epsilon r\cos\theta \\
r(1+\epsilon\cos\theta) &= \epsilon k \\
r &= \frac{\epsilon k}{1+\epsilon\cos\theta}
\end{aligned}
$$

Thus, our result from the previous section defines a conic (in polar coordinates). Further, we know that the eccentricity ϵ is $\sqrt{1+\frac{2El^2}{\mu k^2}}$. Now, with a conic, an eccentricity of less than one is an ellipse (with 0 being a circle), and eccentricity of 1 is a parabola, and a larger eccentricity is a hyperbola. But if we're moving in a parabola, we know that $\sqrt{1+\frac{2El^2}{\mu k^2}}$ is one, so we know that $\frac{2El^2}{\mu k^2}$ is zero, so E must be zero. Thus, if the two objects move in a parabola, the total energy is zero. This makes sense, because this would be the case with two stationary objects infinitely far apart—the conditions of escape velocity.

8 A Simple Lagrangian Mechanics Problem

Finally, let's try our hand at a simple problem that you might actually be expected to solve. Suppose you have a long, straight physics wire (massless, frictionless, etc.), which is fixed at the origin and rotating around the origin at ω. Furthermore, some external agent keeps this rotation always constant. Now, on this wire, there is a mass (it is strung on, so it can't fall off) which can slide up and down freely, which has mass m and is thus acted upon by gravity. The mass begins at position q_0 with velocity v_0. Assuming that the wire starts horizontally, what are the equations of motion for the mass?

What ω, for a given q_0 and v_0, does not allow the mass to escape to infinity? If the mass does not escape, what area of the wire does it cover?

Now, it seems natural to do this problem in polar coordinates, so let's begin writing down the energies. $T = \frac{1}{2}m?$. Hmmm. What is the kinetic energy? Well, we can figure it out by changing to rectangular coordinates momentarily. There, $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$, so since $x = r\cos\theta$, $y = r\sin\theta$, we see that:

$$\begin{aligned}
\dot{x} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\
\dot{y} &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta \\
\dot{x}^2 + \dot{y}^2 &= \dot{r}^2 + r^2\dot{\theta}^2
\end{aligned}$$

So $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$. This, by the way, is a formula to remember. We actually used it before, but didn't prove it. Now we did. Anyway, $U = mgr\sin\theta = mgr\sin\omega t$. We know that $\dot{\theta}$ is just ω, so we know that $L = \frac{1}{2}m(\dot{r}^2 + r^2\omega^2) - mgr\sin\omega t$. Now, we can use Lagrange's Law:

$$\begin{aligned}
\frac{\partial L}{\partial r} &= mr\omega^2 - mg\sin\omega t \\
\frac{\partial L}{\partial \dot{r}} &= m\dot{r} \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} &= m\ddot{r}
\end{aligned}$$

Cancelling and equating, we get

$$\ddot{r} = \omega^2 r - g\sin\omega t$$

So let's go about solving the differential equation. Now, the only place $\sin\omega t$ could have come from is from a $\sin\omega t$, because sines stay sines under differentiation. So, we let $r = \bar{r} + A\sin\omega t$. Then,

$$\begin{aligned}
\ddot{r} &= \ddot{\bar{r}} - \omega^2 A\sin\omega t \\
&= \omega^2 r - g\sin\omega t \\
&= \omega^2\bar{r} + \omega^2 Asin(\omega t) - g\sin\omega t
\end{aligned}$$

Now, we would really like to solve the equation $\ddot{\bar{r}} = \omega^2 \bar{r}$, so we assume that that is indeed the equation we should solve. To do that, we need the two $\sin \omega t$ terms to vanish, so $A = \frac{g}{2\omega^2}$. This leaves us with:

$$\begin{aligned}
\ddot{\bar{r}} &= \omega^2 \bar{r} \\
\bar{r} &= Ce^{-\omega t} + De^{\omega t} \\
\bar{r} &= G \cosh \omega t + H \sinh \omega t
\end{aligned}$$

Since we know that at $t = 0$, the position is q_0 and velocity is v_0, we plug these conditions in, and get $G = q_0$, $H = \frac{1}{\omega} v_0 - \frac{g}{2\omega^2}$. Thus, our final equation of motion is

$$r = q_0 \cosh \omega t + \left(\frac{v_0}{\omega} - \frac{g}{2\omega^2} \right) \sinh \omega t + \frac{g}{2\omega^2} \sin \omega t$$

We can investigate this pretty equation further. If $A \neq B$, r will rapidly go to infinity, as both sinh and cosh approximate $\frac{e^x}{2}$. Thus, we need to start it so that $q_0 = \frac{g}{2\omega^2} - \frac{v_0}{\omega}$. We can, of course, solve this for ω if we want. In the special case where $v_0 = 0$, the equation of motion is

$$r = \frac{g}{2\omega^2} (e^{-\omega t} + \sin \omega t)$$

. If we graph this, we find that, after a short while, this reaches a circle standing vertically, with the lowest point at the origin.

9 Where to Go From Here

If you're interested, we'd suggest looking up "Lagrangian Mechanics Problems" on google, which will give you plenty of interesting problems to try your hand. Its important to do problems, because they often teach you tricks. If you want to look up interesting things to do with Lagrangian mechanics, I would also suggest looking up "Lagrangian Mechanics Wick Rotation". This is a method to transform a dynamics problem with moving objects into a statics problem with springs by making time imaginary. Its really cool. There are also several physics textbooks with Lagrangian Mechanics available for free online. Finally, you may want to take a look at Hamiltonian mechanics, which is an abstraction on top of Lagrangian mechanics which is, surprisingly enough, identical to optics, thus combining two very distinct fields.